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The swimming of flexible slender bodies in waves

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In this paper slender-body theory is applied to flexible bodies. The bodies are assumed to have a constant forward velocity normal to the crests of a regular train of two-dimensional waves and to move at a nearly constant depth. A flexible recoil mode is defined which is the dynamic counterpart of the stretched straight fish position defined by Lighthill (1960) for uniform flow conditions. Expressions are derived for the mean thrust and for the mean rate of working, and permit the evaluation of conditions for efficient propulsion. By properly adapting the motions of the body to the oncoming waves, energy can be extracted from the waves in such a way that it can be used for propulsion. This phenomenon may help to explain the high speeds that cetacea are observed to sustain over long periods of time.

1. Introduction

Lighthill (1960) worked out a theory for the swimming of a slender fish through water at rest, i.e. in uniform flow conditions. Several refinements and extensions have been made since that time. Lighthill's original formulation, however, appears to be well suited as a starting point for the statement of the swimming problem in waves. In this paper an attempt is made to get some insight into the adaptation of swimming movements to an oncoming regular train of waves. Head seas as well as following seas with the crests of the waves perpendicular to the direction of motion of the body are included. The case where the following sea has the same velocity as the body will not be discussed here. This case is analogous to the case of bow-wave riding treated by Focke (1965).

The boundary-value problems that will arise for the velocity potential of the flow field outside the boundary layer can be cast in a form which is formally equivalent to that of problems arising in the uniform flow case. In the evaluation of the forces exerted on the body, the rate of working by the body and the energy extracted from the waves, however, some new features will appear. In particular, the effects of the adaptation on the efficiency of the propulsion are discussed. The propulsive force is assumed to balance, in the mean, the viscous drag and the wave drag. The fluctuations in the resultant of these forces are assumed to be such that the forward speed of the body can be treated as a constant. The slender bodies discussed in this paper are assumed to exhibit left–

right symmetry with the lateral displacements parallel to the plane of symmetry, which is typical of cetacea. The fact that their fins and tails are not slender in the sense used in this paper is not expected to affect severely the more qualitative conclusions related to their propulsive efficiency in the non-uniform flows that occur near a wavy sea surface.

2. The velocity field

The Cartesian co-ordinate system (x, y, z) performs a steady translation with the body in the $-x$ direction at velocity U. The plane $z = 0$ is at a constant depth d below the calm free surface of the water. The free surface is disturbed by a regular train of two-dimensional waves of wavelength λ, height a and phase velocity $U+c$ with respect to the (x, y, z) system in the $+x$ direction; c is positive in a head sea and negative in a following sea.

The velocity potential for these waves is given by

$$\phi_w(x, z, t) = ac\exp[2\pi\lambda^{-1}(z-d)]\cos\{2\pi\lambda^{-1}[x-(U+c)t]\}, \tag{2.1}$$

while the amplitude of the waves at the swimming depth d is given by

$$a\exp[2\pi(-d)/\lambda] = a^*. \tag{2.2}$$

To formulate the boundary conditions at the body surface we follow Lighthill (1960). The body is considered to be 'stretched straight' when, in uniform flow and without free-surface effects, no resultant normal force acts on any cross-section. The velocity potential in this case is expressed as

$$Ux + \Phi_0(x, y, z). \tag{2.3}$$

We suppose that the cross-sections of the body perform displacements $h(x,t)$ in the z direction without altering their shapes. $y = 0$ is a plane of symmetry. If the stretched straight body is given by $F(x, y, z) = 0$, the moving body is given by $F(X, Y, Z) = 0$ after introduction of the co-ordinates (X, Y, Z, T), where

$$X = x, \quad Y = y, \quad T = t, \quad Z = z - h(x,t). \tag{2.4}$$

In these co-ordinates the velocity potential Φ satisfies the transformed Laplace equation and can be decomposed as follows:

$$\Phi(X, Y, Z, T) = UX + \Phi_0(X, Y, Z) + \Phi_1(X, Y, Z, T) + \Phi_w(X, Z, T) + \Phi_2(X, Y, Z, T). \tag{2.5}$$

In (2.5) the function Φ_0 is the same function as in (2.3). Φ_1 is the perturbation potential due to the displacements h of the body in the absence of waves. Φ_w is the potential of the oncoming waves, which follows from (2.1) with (2.4):

$$\phi_w(x, z, t) = \Phi_w(X, Z, T). \tag{2.6}$$

Φ_2 is the perturbation potential due to the waves around the constrained body

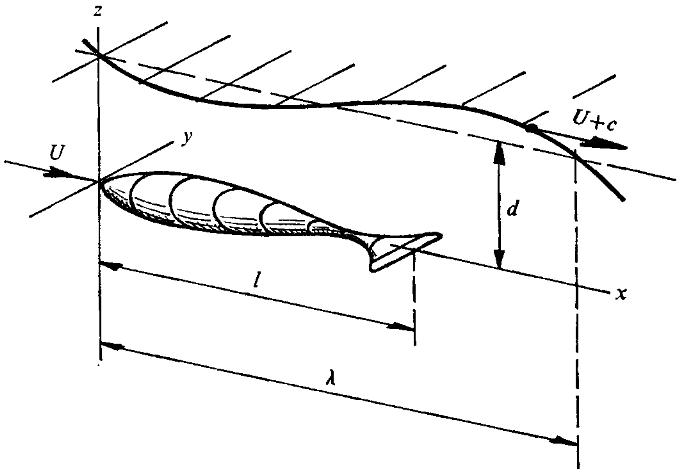

FIGURE 1

($h = 0$). Corrections related to the presence of the free surface have not been made. The boundary condition at the body surface is obtained by putting

$$D\{F(X, Y, Z)\}/Dt,$$

the derivative following a particle of water, equal to zero:

$$-\frac{\partial h}{\partial T}\frac{\partial F}{\partial Z}+\left(\frac{\partial \Phi}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial \Phi}{\partial Z}\right)\left(\frac{\partial F}{\partial X}-\frac{\partial h}{\partial X}\frac{\partial F}{\partial Z}\right)+\frac{\partial \Phi}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \Phi}{\partial Z}\frac{\partial F}{\partial Z}=0. \tag{2.7}$$

For a slender body $\partial F/\partial X$ is small compared with $\partial F/\partial Y$ and $\partial F/\partial Z$. Derivatives of Φ_0, Φ_1, Φ_w and Φ_2 with respect to X, Y and Z are small compared with U. Moreover $\partial h/\partial X$ is assumed to be small while $\partial h/\partial T$ is assumed to be small with respect to U. Omitting products of small quantities one finds as the boundary condition for Φ_0

$$U\frac{\partial F}{\partial X}+\frac{\partial \Phi_0}{\partial Y}\frac{\partial F}{\partial Y}+\frac{\partial \Phi_0}{\partial Z}\frac{\partial F}{\partial Z}=0 \quad \text{at} \quad F(X, Y, Z)=0, \tag{2.8}$$

and with $\psi = \Phi_1 + \Phi_2$ one has for ψ

$$\frac{\partial \psi}{\partial Y}\frac{\partial F}{\partial Y}+\left(\frac{\partial \psi}{\partial Z}+\frac{\partial \Phi_w}{\partial Z}-\frac{\partial h}{\partial T}-U\frac{\partial h}{\partial X}\right)=0 \quad \text{at} \quad F(X, Y, Z)=0. \tag{2.9}$$

From (2.1) and (2.2) one finds, using (2.4),

$$\partial \Phi_w/\partial Z = \Phi^*_{wZ}\exp\{2\pi\lambda^{-1}[Z+h(X,T)]\}, \tag{2.10}$$

with

$$\Phi^*_{wZ} = \alpha^*\cos\{2\pi\lambda^{-1}[X-(U+c)\,T]\} \tag{2.11}$$

and

$$\alpha^* = ac\frac{2\pi}{\lambda}\exp\left(\frac{2\pi}{\lambda}(-d)\right) = a^*\left(\frac{2\pi c}{\lambda}\right). \tag{2.12}$$

Thus, if we assume that the lateral dimensions of the body and the displacements are small with respect to the wavelength λ also, we can write as the boundary condition for ψ

$$\frac{\partial\psi}{\partial Y}\frac{\partial F}{\partial Y}+\left(\frac{\partial\psi}{\partial Z}-w^*\right)\frac{\partial\psi}{\partial Z}=0 \quad \text{at} \quad F(X,Y,Z)=0, \tag{2.13}$$

where
$$w^*=\frac{\partial h}{\partial T}+U\frac{\partial h}{\partial X}-\Phi^*_{wZ}=w-\Phi^*_{wZ}. \tag{2.14}$$

For each X, ψ is the potential of the two-dimensional flow in the Y,Z plane resulting from the movement of an infinite cylinder C_X with cross-section $S(X)$ with a velocity w^* in the Z direction through an unbounded water mass. Then, with

$$\psi(X,Y,Z,T)=w^*(X,T)\,\phi(Y,Z;X), \tag{2.15}$$

ϕ satisfies
$$\partial^2\phi/\partial Y^2+\partial^2\phi/\partial Z^2=0, \tag{2.16}$$

and the boundary condition $\phi\to 0$ at infinity.

3. The hydrodynamic forces on the body

The hydrostatic lift forces are assumed to balance the weight of a slice of the body between any two vertical cross-sections, so that these terms can be omitted from the equations. The wavelength λ and the body length l are assumed to be of the same order of magnitude. The dimensions of the body in the Y and Z directions, the displacements h and the amplitude of the waves are assumed to be small with respect to the body length l, say of order ϵl, where ϵ is a small parameter. The derivatives of Φ_0, Φ_1 and Φ_2 with respect to Y and Z as well as $\partial\Phi_w/\partial X$ and $\partial\phi_w/\partial Z$ are taken to be of order ϵU. The derivatives of Φ_0, Φ_1 and Φ_2 with respect to X are of order $U\epsilon^2\log\epsilon$. Then, by retaining terms up to and including those of orders ϵ^2U^2 and $\epsilon^2U^2\log\epsilon$, one finds from Bernoulli's equation

$$p=p_0+p_1+p_2+p_3, \tag{3.1}$$

with the following decomposition:

$$\left.\begin{aligned}
p_0&=\text{constant}-\rho U\frac{\partial\Phi_0}{\partial X}-\tfrac{1}{2}\rho\left\{\left(\frac{\partial\Phi_0}{\partial Y}\right)^2+\left(\frac{\partial\Phi_0}{\partial Z}\right)^2\right\},\\
p_1&=-\rho\left\{\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\psi+\frac{\partial\Phi_0}{\partial Y}\frac{\partial\psi}{\partial Y}+\frac{\partial\Phi_0}{\partial Z}\left(\frac{\partial\psi}{\partial Z}+\frac{\partial\Phi_w}{\partial Z}-w\right)\right\},\\
p_2&=\rho\left\{-\left(\frac{\partial}{\partial T}+U\frac{\partial}{\partial X}\right)\Phi_w+w\frac{\partial\Phi_w}{\partial Z}-\tfrac{1}{2}\left(\frac{\partial\Phi_w}{\partial X}\right)^2-\tfrac{1}{2}\left(\frac{\partial\Phi_w}{\partial Z}\right)^2\right\},\\
p_3&=\rho\left\{\left(w-\frac{\partial\Phi_w}{\partial Z}\right)\frac{\partial\psi}{\partial Z}-\tfrac{1}{2}\left(\frac{\partial\psi}{\partial Y}\right)^2-\tfrac{1}{2}\left(\frac{\partial\psi}{\partial Z}\right)^2\right\}.
\end{aligned}\right\} \tag{3.2}$$

We first calculate the lift force $L(X,T)$ per unit length. Owing to the symmetry with respect to the plane $y=0$, this lift force acts in the plane $y=0$ and introduces no torsional moments. The part p_0 is the pressure distribution associated with the stretched straight position and, by definition, does not contribute to the lift force. The first term of p_1 leads to a contribution of order ϵ^3 to $L(X,T)$:

$$L_1 = -\rho \oint_X (\partial/\partial T + U\,\partial/\partial X)\,\psi\, dY. \tag{3.3}$$

Defining $A(X)$ by

$$A(X) = \oint_X \phi\, dY, \tag{3.4}$$

where $\rho A(X)$ is the virtual mass per unit length of a cylinder C_X moving in the Z direction and ϕ is defined by (2.15), the contribution L_1 can be expressed as

$$L_1 = -\rho(\partial/\partial T + U\,\partial/\partial X)\{w^* A(X)\}. \tag{3.5}$$

The second contribution of order ϵ^3 to the lift follows from the first term of p_2:

$$L_2 = -\rho \oint_X (\partial/\partial T + U\,\partial/\partial X)\,\Phi_w\, dY. \tag{3.6}$$

With $\Phi_w(X,Z,T) = \Phi_w(X,O,T) + Z\Phi_{wZ}(X,O,T) + \ldots$, one finds, to order ϵ^3,

$$L_2 = \rho S(X)\,(\partial/\partial T + U\,\partial/\partial X)\,\Phi^*_{wZ}, \tag{3.7}$$

where $S(X)$ is the area of the cross-section. The remaining terms in (3.2) contribute to higher-order terms in the lift only. Combining (3.5) and (3.7) we thus obtain for the lift, to order ϵ^3,

$$L(X,T) = -\rho\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\{w^* A(X)\} + \rho S(X)\left(\frac{\partial}{\partial T} + U\frac{\partial}{\partial X}\right)\Phi^*_{wZ}. \tag{3.8}$$

Without waves, $L(X,T)$ is independent of the area $S(X)$ of the cross-section and (3.8) reduces to Lighthill's (1960) result. It may be noted that, as Lighthill (1960) showed, the error in (3.8) in the case of uniform flow is only of order ϵ^5. The error in the terms of (3.8) due to the presence of the waves is of order ϵ^4. Changing the order of the differentiations in the second term on the right-hand side of (3.8) shows that this term results from the vertical component of the pressure gradient due to the waves.

The displacements of an unconstrained body cannot be chosen arbitrarily. First, the time rate of change of the momentum of the body in the z direction must be equal to the resultant of the lift forces. Second, the time rate of change of the angular momentum of the body about the y axis must be equal to the moment of the lift forces about that axis:

$$\left.\begin{aligned}\rho\int_0^l S(x)\frac{\partial^2 h}{\partial t^2}\,dx &= \int_0^l L(x,t)\,dx,\\ \rho\int_0^l xS(x)\frac{\partial^2 h}{\partial t^2}\,dx &= \int_0^l xL(x,t)\,dx.\end{aligned}\right\} \tag{3.9}$$

These equations will be discussed in some detail in the next section.

The mean thrust $\overline{T}$ is evaluated in the appendix. The result, to order ϵ^4, is

$$\overline{T} = -\rho\int_0^l \overline{\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\{w^*A(x)\}}\,dx+\tfrac{1}{2}\rho\int_0^l \overline{w^{*2}}\frac{dA}{dx}\,dx$$
$$+\rho\int_0^l S(x)\overline{\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial x}\right)\phi^*_{wz}}\,dx-\rho\int_0^l \overline{w\,\phi^*_{wz}}\frac{\partial S}{dx}dx. \quad (3.10)$$

It should be observed that the case $U+c = 0$ has been excluded in the derivation of (3.10). Both $h(x,t)$ and ϕ_w have been assumed to be oscillatory. At the nose of the body we put $S(0) = A(0) = 0$. At the tail we assume $S(l) = 0$ but $A(l) \neq 0$. Then $\overline{T}$ can be expressed as

$$\overline{T} = \tfrac{1}{2}\rho A(l)\overline{\left[\left(\frac{\partial h}{\partial t}-\phi^*_{wz}\right)^2 - U^2\left(\frac{\partial h}{\partial x}\right)^2\right]_{x=l}} + \rho\int_0^l (A+S)\,\overline{w\phi^*_{wzx}}dx. \quad (3.11)$$

Without waves ($\phi^*_{wz} = 0$) the mean thrust is determined by the movements of the tail only, as in Lighthill's (1960) result.

It may be observed here that for a constrained rigid body

$$\overline{T} = \tfrac{1}{2}\rho A(l)\,\overline{\{(\phi^*_{wz})^2 - U^2(\partial h/\partial x)^2_{x=l}\}}. \quad (3.12)$$

Those who are familiar with slender-wing theory will recognize the second term on the right-hand side of (3.12) as the vortex drag of a span loading of elliptic form.

The mean rate of working by the body is given by

$$\overline{W} = -\int_0^l \overline{\frac{\partial h}{\partial t}L(x,t)}\,dx. \quad (3.13)$$

Using (3.8) with, again, $S(0) = A(0) = 0$, $S(l) = 0$ and $A(l) \neq 0$ one has

$$\overline{W} = \rho A(l)\,U\overline{\left[\frac{\partial h}{\partial t}w^*\right]_{x=l}} + \rho\int_0^l (A+S)\overline{\frac{\partial w}{\partial t}\phi^*_{wz}}dx + \rho U\int_0^l \overline{\phi^*_{wz}\frac{\partial h}{\partial t}}\frac{dS}{dx}dx. \quad (3.14)$$

As usual, the quality of the propulsion is measured by the quantity

$$\eta = \overline{T}U/\overline{W}. \quad (3.15)$$

A swimming problem may now be formulated as follows: find physiologically plausible solutions $h(x,t)$ of the equations of motion (3.9) giving a positive and sufficiently large $\overline{T}$ at a given U at high η. In a calm sea without waves, η cannot exceed unity and to generate a positive $\overline{T}$ the rate of working $\overline{W}$ must be positive. In the presence of waves η may exceed unity and $\overline{W}$ is not necessarily positive when $\overline{T}$ is positive. Before discussing these aspects in more detail we return to the equations of motion (3.9).

4. The flexible recoil mode

An interesting particular solution $\tilde{h}(x,t)$ of the equations of motion is obtained by equating the local time rate of change of the lateral momentum of the body and the local lateral force exerted on the body by the time-varying part of the pressure of the water:

$$\rho S(x)\,\partial^2\tilde{h}/\partial t^2 = \tilde{L}(x,t). \tag{4.1}$$

It can be shown that, if ϕ^*_{wz} is given by

$$\phi^*_{wz} = \alpha^* \cos\{2\pi\lambda^{-1}[x-(U+c)\,t]\}, \tag{4.2}$$

one obtains

$$\tilde{h}(x,t) = \tilde{h}_1(x)\cos(2\pi x/\lambda - \omega t) + \tilde{h}_2(x)\sin(2\pi x/\lambda - \omega t), \tag{4.3}$$

where

$$\omega/2\pi = |U+c|/\lambda \tag{4.4}$$

is the frequency of encounter. The functions $\tilde{h}_1(x)$ and $\tilde{h}_2(x)$ are given in the appendix.

This flexible recoil mode may be considered as the dynamic counterpart of the stretched straight position ($h = 0$) in uniform flow. The local mean rate of working is equal to zero,

$$-\overline{\frac{\partial\tilde{h}}{\partial t}\tilde{L}(x,t)} = -\rho S(x)\,\tfrac{1}{2}\overline{\frac{\partial}{\partial t}\left(\frac{\partial\tilde{h}}{\partial t}\right)^2} = 0, \tag{4.5}$$

and it is clear that the work done by the body as a whole is also equal to zero. Therefore this easy-going motion would appear to be a natural point of departure for an animal trying to minimize effort. In order to gain some further insight into the flexible recoil mode we consider some special cases.

(i) Putting $S = 0$ and $A \neq 0$ leads to

$$\tilde{h}_1 = 0, \quad \tilde{h}_2 = \frac{-\alpha^*}{2\pi c/\lambda} = a^*. \tag{4.6}$$

From (4.2) and (4.3) one finds for this case

$$\tilde{w} = \partial\tilde{h}/\partial t + U\,\partial\tilde{h}/\partial x = \phi^*_{wz}, \tag{4.7}$$

which implies that there is no cross-flow. It follows, moreover, that the tail of a body which is characterized by $S \ll A$ and $S \to 0$ for $x \to l$ will tend to follow the oncoming flow smoothly.

(ii) Consider a body which satisfies the following symmetry relations:

$$S(x) = S(l-x), \quad A(x) = A(l-x). \tag{4.8}$$

It may be observed that a body which is symmetric with respect to the midship position $x = \frac{1}{2}l$ satisfies (4.8). Using (4.8) one obtains from (A 9) (see appendix)

$$\tilde{h}_1(x) = -\tilde{h}_1(l-x), \quad \tilde{h}_2(x) = \tilde{h}_2(l-x). \tag{4.9}$$

By putting

$$\tilde{w}(x,t) = \frac{\partial\tilde{h}}{\partial t} + U\frac{\partial\tilde{h}}{\partial x} = \tilde{w}_1(x)\cos\left(\frac{2\pi x}{\lambda} - \omega t\right) + \tilde{w}_2(x)\sin\left(\frac{2\pi x}{\lambda} - \omega t\right), \tag{4.10}$$

one obtains, using (4.3) and (4.9),

$$\tilde{w}_1(x) = \tilde{w}_1(l-x), \quad \tilde{w}_2(x) = -\tilde{w}_2(l-x). \tag{4.11}$$

If, in addition to (4.8), one assumes $S(0) = S(l) = A(0) = A(l) = 0$, it follows from (3.11) that the mean thrust $\overline{T}$ is equal to zero. Furthermore, it may be noticed that for a body with $A \approx S$ which near the nose is characterized by $A \doteqdot x^n\,(n > 0)$ and at the afterbody by $A \doteqdot (l-x)^n\,(n > 0)$ one finds from (A 9) that $\tilde{h}_1(x) \rightarrow 0$ and $\tilde{h}_2(x) \rightarrow a^*$ for both $x \rightarrow 0$ and $x \rightarrow l$. The nose, as well as the afterbody, tends to follow the oncoming flow smoothly and the cross-flow $\tilde{w}^*$ tends to zero both as $x \rightarrow 0$ and as $x \rightarrow l$. It is clear that to such a body one may attach an inoperative thin tail without affecting the flexible recoil mode of the body. In view of possible extensions of the present formulation it is interesting to note that such an inoperative thin tail need not be slender in the sense used here. At present, however, it is more appropriate to remain within the scope of the slender-body formulation, and we propose to discuss swimming motions of a slender body which can be characterized as follows:

$$\left.\begin{array}{c} A \approx S \quad \text{except at the tail, where} \quad A \gg S, \\ S(x) = S(l-x), \quad S(0) = 0, \quad S(l) = 0, \\ A(x) = A(l-x) \quad \text{except near} \quad x = 0, l, \quad A(0) = 0, \quad A(l) \neq 0. \end{array}\right\} \tag{4.12}$$

5. The adaptation of the swimming movements to the oncoming waves

The displacements $h(x,t)$ satisfying the equations of motion (3.9) can be decomposed as follows:

$$h(x,t) = f(x,t) + \tilde{h}(x,t), \tag{5.1}$$

where $\tilde{h}(x,t)$ is the flexible recoil mode and $f(x,t)$ is a solution of the homogeneous part of (3.9), which is obtained by putting $\phi^*_{wz} = 0$, as in uniform oncoming flow.

The swimming problem can now be stated as the problem of adapting the voluntary and active displacements f to the passive displacements $\tilde{h}$. On the assumption that the situation at the tail dominates the generation of thrust it is natural to start with the evaluation of the first terms in (3.11) and (3.14). For a body of the type (4.12), with $S(l) = 0$ and $A(l) \neq 0$, one has at $x = l$

$$\partial\tilde{h}/\partial t + U\,\partial\tilde{h}/\partial x = \phi^*_{wz},$$

and one finds upon substitution of (5.1) into these terms

$$\overline{T} \approx \tfrac{1}{2}\rho A(l)\overline{\left[\left(\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial t} - U\frac{\partial f}{\partial x} - 2U\frac{\partial \tilde{h}}{\partial x}\right)\right]_{x=l}}, \tag{5.2}$$

$$\overline{W} \approx \rho U A(l)\overline{\left[\left(\frac{\partial f}{\partial t} + \frac{\partial \tilde{h}}{\partial t}\right)\left(\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x}\right)\right]_{x=l}}, \tag{5.3}$$

$$\eta \approx \tfrac{1}{2}\frac{\overline{\left[\left(\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x}\right)\left(\frac{\partial f}{\partial t} - U\frac{\partial f}{\partial x} - 2U\frac{\partial \tilde{h}}{\partial x}\right)\right]_{x=l}}}{\overline{\left\{\left(\frac{\partial f}{\partial t} + \frac{\partial \tilde{h}}{\partial t}\right)\left(\frac{\partial f}{\partial t} + U\frac{\partial f}{\partial x}\right)\right\}_{x=l}}}. \tag{5.4}$$

In the flexible recoil mode the vertical displacements $\tilde{h}$ are oscillations with the frequency of encounter (4.4). If the voluntary displacements f were characterized by a different frequency, the mean of the cross-terms involving both $\tilde{h}$ and f would vanish. There would be no correlation and the interference drag or thrust as well as the contribution of these cross-terms to $\overline{W}$ would be equal to zero. No benefit could be obtained from the waves and the swimming problem would be essentially the same as in uniform flow conditions.

From (5.2) and (5.3) one obtains

$$\overline{W} - \overline{T}U \approx \tfrac{1}{2}\rho A(l)\, U[\overline{w^{*2}} + \overline{2w^*\phi^*_{wz}}]_{x=l}, \tag{5.5a}$$

or alternatively

$$\overline{W} - \overline{T}U \approx \tfrac{1}{2}\rho A(l)\, U[\overline{w^2} - \overline{\phi^{*2}_{wz}}]_{x=l}. \tag{5.5b}$$

Without waves one has $w = w^*$ and the right-hand sides of (5.5) can be interpreted as the mean rate of shedding of kinetic energy into the wake. It is clear from (5.5a) that the mean rate of wasting energy can be reduced by correlating w^* and ϕ^*_{wz} negatively at $x = l$. Moreover, the terms in brackets have a minimum value of $-\overline{\phi^{*2}_{wz}}$ for

$$w^* = -\phi^*_{wz} \quad \text{at} \quad x = l, \tag{5.6a}$$

which can also be expressed as

$$w = 0 \quad \text{at} \quad x = l. \tag{5.6b}$$

For motions of the tail satisfying (5.6) the resultant cross-flow w^* at the tail is exactly minus the local vertical component of the orbital velocity of the surrounding water particles due to the waves. The vertical component of the velocity of the water has vanished just above and just below the tail. The maximum mean rate at which the body can extract energy from the waves is estimated as minus the right-hand sides of (5.5) after substitution of (5.6):

$$\overline{(\partial E/\partial t)}_{\max} \approx \overline{T}^*U, \tag{5.7}$$

where $\overline{T}^*$ is a reference thrust defined by

$$\overline{T}^* = \tfrac{1}{2}\rho A(l)\,\overline{\phi^{*2}_{wz}}. \tag{5.8}$$

From the volume which is effectively being swept by the tail over long periods of time, approximately half of the kinetic energy present due to the waves can be extracted and made available for propulsion.

From (5.5) and (5.7) it follows that the mean rate at which energy is being 'wasted' can be estimated as

$$\text{waste} \approx \tfrac{1}{2}\rho A(l)\,\overline{(w^* + \phi^*_{wz})^2_{x=l}} = \tfrac{1}{2}\rho A(l)\,\overline{w^2_{x=l}}. \tag{5.9}$$

It should be observed that, in this 'waste', the difference between the maximum mean rate at which energy could be extracted from the waves and the mean rate at which it is actually being extracted has been included. This is evident in the power balance:

$$\overline{W} + \overline{(\partial E/\partial t)}_{\max} = \overline{T}U + \text{waste}. \tag{5.10}$$

In order to discuss the swimming problem in more detail it is convenient to modify our notation. After an appropriate shift of the origin of the time axis one may put

$$\left.\begin{aligned}\left[\frac{\partial \tilde{h}}{\partial t}\right]_{x=l} &= \alpha^*\left(1+\frac{U}{c}\right)\cos\omega t = \alpha\cos\omega t,\\ \left[U\frac{\partial \tilde{h}}{\partial x}\right]_{x=l} &= \alpha^*\left(-\frac{U}{c}\right)\cos\omega t = \beta\cos\omega t.\end{aligned}\right\} \quad (5.11)$$

Assuming that the voluntary motions of the tail are perfectly correlated with the waves one may write

$$[\partial f/\partial t]_{x=l} = \gamma\cos\omega t, \quad [U\,\partial f/\partial x]_{x=l} = \delta\cos\omega t. \quad (5.12)$$

Substitution of (5.11) and (5.12) into (5.2)–(5.4) yields

$$\overline{T} \approx \tfrac{1}{4}\rho A(l)\{(\gamma+\delta)(\gamma-\delta-2\beta)\}, \quad (5.13a)$$

$$\overline{W} \approx \tfrac{1}{2}\rho A(l)\,U\{(\gamma+\delta)(\alpha+\gamma)\}, \quad (5.13b)$$

and with $\gamma+\delta \neq 0$,

$$\eta \approx \tfrac{1}{2}(\gamma-\delta-2\beta)/(\alpha+\gamma). \quad (5.13c)$$

Adopting (5.6), tentatively, as a condition to be satisfied by the voluntary motions of the tail leads to

$$\gamma+\delta = -\alpha^*. \quad (5.14)$$

Using (5.14) and introducing a parameter ν defined by

$$\delta = (U/c+\tfrac{1}{2}\nu)\,\alpha^* \quad (5.15)$$

gives

$$\gamma = -(1+U/c+\tfrac{1}{2}\nu)\,\alpha^*. \quad (5.16)$$

Using (5.8) then makes it possible to write

$$\overline{T} \approx \overline{T}^*(1+\nu), \quad \overline{W} \approx \nu\overline{T}^*U, \quad \eta \approx 1+1/\nu. \quad (5.17)$$

The mean rate at which the body does work is proportional to ν while an increase in ν is accompanied by an increase in the mean thrust. Whenever $\overline{T}^*$ is insufficient to sustain a speed U the body must do work and ν must be positive.

In order to describe the movements of the tail it is helpful to introduce two co-ordinate systems.

(i) The system (x_1, y_1, z_1, t_1), where

$$x_1 = x-Ut, \quad y_1 = y, z_1 = z, t_1 = t, \quad (5.18)$$

which is fixed to the water at rest far from the surface.

(ii) The system (x_2, y_2, z_2, t_2), where

$$x_2 = x-(U+c)\,t, \quad y_2 = y, \quad z_2 = z, \quad t_2 = t, \quad (5.19)$$

which moves steadily with the waves.

In the x, z plane the path of the tail is a vertical line at $x = l$. In the extraction modes satisfying (5.15) and (5.16) this path is symmetrical with respect to $z = 0$. The displacement of the tail is given by

$$z = -\frac{\nu a^*}{2}\frac{c}{U+c}\sin\left(\frac{2\pi}{\lambda}(U+c)\,t\right). \quad (5.20)$$

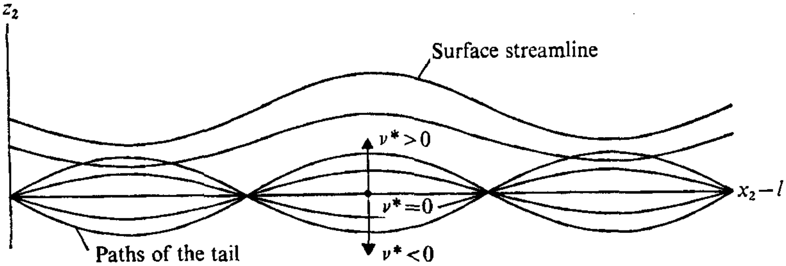

FIGURE 2. The paths of the tail in the x_2, z_2 plane in the extraction modes.

In the x_1, z_1 plane the path of the tail is given by

$$z_1 = \frac{\nu a^*}{2}\frac{c}{U+c}\sin\left[\frac{2\pi}{\lambda}\left(\frac{U+c}{U}\right)(x_1-l)\right]. \tag{5.21}$$

The 'wavelength' of this path is

$$\mu = \lambda U/|U+c|. \tag{5.22}$$

Moreover it follows from (5.6*b*) that in the x_1, z_1 plane the tail is tangential to its path. In the x_2, z_2 plane the path of tail becomes

$$z_2 = \frac{\nu a^*}{2}\frac{c}{U+c}\sin\left(\frac{2\pi}{\lambda}(x_2-l)\right). \tag{5.23}$$

The 'wavelength' of this path is λ.

In the x_2, z_2 plane the waves induce a steady flow field and at the swimming depth one obtains a streamline

$$\tilde{z}_2 = -a^*\sin\left[2\pi\lambda^{-1}(x_2-l)\right]. \tag{5.24}$$

Comparison of (5.23) and (5.24) reveals that the path of the tail in the x_2, z_2 plane can be obtained from the streamlines due to the waves by a simple multiplication:

$$z_2 = \nu^*\tilde{z}_2, \quad \text{with} \quad \nu^* = -\frac{\nu}{2}\left(\frac{c}{U+c}\right). \tag{5.25}$$

In the special case $\nu = 0$, the path of the tail is a straight line and the slope of the tail is equal to zero. In the special case $z_2 = \tilde{z}_2$, the path of the tail coincides with the streamlines (5.24) but, in contrast to the situation in the flexible recoil mode, the tail is not tangential to its path in the x_2, z_2 plane. For $\nu^* < a/a^*$ the tail comes closest to the surface of the water at the troughs of the waves. For $\nu^* > a/a^*$ this happens at the crests of the waves.

The voluntary movements of the tail in the extraction modes are given by

$$\left.\begin{aligned} [\partial f/\partial t]_{x=l} &= \alpha^*(U/c+\tfrac{1}{2}\nu)\cos\omega t, \\ [U\,\partial f/\partial x]_{x=l} &= -\alpha^*(1+U/c+\tfrac{1}{2}\nu)\cos\omega t. \end{aligned}\right\} \tag{5.26}$$

$[\partial f/\partial t]_{x=l}$ and $[U\,\partial f/\partial x]_{x=l}$ are correlated negatively for $\frac{1}{2}\nu > -U/c$ and for $\frac{1}{2}\nu < -1-U/c$. In contrast to the situation in uniform flow conditions, there are cases with positive correlations, for $-U/c-1 < \frac{1}{2}\nu < -U/c$, where the propulsion may be satisfactory. Lighthill (1960, 1970) showed that in uniform flow conditions the function $f(x,t)$ can take the form of a wave passing down the body at a phase velocity somewhat larger than U. If such an $f(x,t)$ does not satisfy the homogeneous part of the equations of motion (3.9) it is necessary to add some terms, e.g. a 'rigid recoil' displacement, in order to obtain a homogeneous solution. In the present, non-uniform case $f(x,t)$ should be characterized by the frequency of encounter. The movements of the tail required in the extraction modes can be brought about by, say,

$$f(x,t) = g(x)\cos\frac{2\pi}{k}\{x-(U+v)t\}+(g_1+g_2x)\sin\frac{2\pi}{k}(U+v)t + (g_3+g_4x)\cos\frac{2\pi}{k}(U+v)t, \quad (5.27)$$

with the correlation condition

$$|U+v|/k = |U+c|/\lambda, \quad (5.28)$$

where $g(x)$ is the amplitude of a wave with wavelength k passing along the body at a phase velocity $U+v$. It is clear from (5.28) that the first term on the right-hand side of (5.27) is not necessarily a wave passing down the body and one may construct superpositions for different values of k and v.

It may not always be possible for a swimming body to realize motions of the tail of the type (5.26) and at the same time obtain sufficient thrust in the extraction modes. Obviously ν cannot become too large. It is clear from (5.26) that, in a following sea, this restriction on ν is less severe than in a head sea. Whenever ν cannot be made large enough to obtain sufficient thrust by (5.17), a compromise must be found.

One may accept motions which satisfy (5.14) only approximately but which remain characterized by the frequency of encounter. On the other hand part of the thrust may be obtained from a component f_u of f which is not characterized by the frequency of encounter. Writing f_c for the part of f which is correlated with the waves one has

$$f = f_c+f_u. \quad (5.29)$$

One then obtains

$$\overline{T} = \overline{T}_c+\overline{T}_u, \quad \overline{W} = \overline{W}_c+\overline{W}_u, \quad (5.30)$$

and with $\eta_c = \overline{T}_c U/\overline{W}_c$ and $\eta_u = \overline{T}_u U/\overline{W}_u$, one may put

$$\eta = \frac{(\overline{T}_c+\overline{T}_u)\,U}{\overline{W}_c+\overline{W}_u} = \frac{\eta_c\eta_u\overline{T}}{\eta_c\overline{T}_u+\eta_u\overline{T}_c}. \quad (5.31)$$

Typical values of η_u are, as in uniform flow conditions, somewhat smaller than unity, say 0·8, but η_c is larger than unity and can be much larger. It is clear that η can be made to satisfy

$$\eta_u \leqslant \eta \leqslant \eta_c. \quad (5.32)$$

The expressions derived in this section permit a numerical evaluation of the propulsion of a slender body swimming in waves. For given movements of the tail, such as those associated with the extraction modes, these calculations are straightforward. The problem of the construction of functions describing the displacements of the body as a whole which bring about such favourable motions of the tail has no unique solution. One is free to select 'physiologically plausible' solutions of the equations of motion. In numerical evaluations it is desirable to verify whether the integrals in the complete expressions for $\overline{T}$ and $\overline{W}$ indeed do not affect the propulsion. It is clear that their effect is not necessarily unfavourable. In cases where these integrals turn out to be significant it may become attractive to allow for phase shifts in the movements of the tail.

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Appendix

The mean thrust

The thrust T can be expressed as the x component of the resultant of the pressure forces at the body surface:

$$T = \iint_S p\,dy\,dz = \iint_S p\,dY\left(\frac{\partial h}{\partial X}dX + dZ\right)$$
$$= \int_0^l L\frac{\partial h}{\partial X}dX + \iint_S p\,dY\,dZ, \qquad \text{(A 1)}$$

where the lift L follows from (3.8) and the pressure p follows from (3.1) and (3.2). It remains to evaluate the last integral in (A 1).

By virtue of d'Alembert's paradox the part p_0 does not contribute to T. Excluding the case $U+c = 0$ and assuming oscillatory h and ϕ_w it is clear that the mean of p_1 over a long time is equal to zero. In p_2 the first term is oscillatory while the term $-\frac{1}{2}\rho\{(\partial\phi_w/\partial X)^2 + (\partial\phi_w/\partial Z)^2\}$ does not depend on X. Therefore the contribution from p_2 to the mean thrust, to order ϵ^4, is given by

$$\iint_S \overline{p_2}\,dY\,dZ = -\rho\int_0^l \overline{w\Phi^*_{wZ}}\frac{dS}{dX}dX. \qquad \text{(A 2)}$$

In p_3 the term $\partial\Phi_w/\partial Z$ is now replaced by Φ^*_{wZ} and using (2.14) one obtains, to order ϵ^4,

$$\iint_S p_3\,dY\,dZ = \rho\iint_S\left\{w^*\frac{\partial\psi}{\partial Z} - \tfrac{1}{2}\left(\frac{\partial\psi}{\partial Y}\right)^2 - \tfrac{1}{2}\left(\frac{\partial\psi}{\partial Z}\right)^2\right\}dY\,dZ. \qquad \text{(A 3)}$$

The integral in (A 3), for a certain value of X, is formally equivalent to the pressure distribution due to the steady motion of a cylinder C_X with constant velocity w^* in the Z direction through water at rest. It should be noticed that p_3 does not involve time derivatives of the potential. Time plays the role of a parameter only. Thus, in order to evaluate (A 3), the following argument applies.

The kinetic energy of the water per unit length is $\frac{1}{2}\rho A w^{*2}$ and the momentum in the Z direction is $\rho A w^*$. A time $\delta X/U$ later the kinetic energy has changed by an amount $\frac{1}{2}\rho(d(Aw^{*2})/dX)\,\delta X$ and the momentum by $\rho(d(Aw^*)/dX)\,\delta X$. In order to bring about this change in momentum the Z component of the force exerted by the body on the water per unit length must do the amount of work $\rho w^*(d(Aw^*)/dX)\,\delta X$. The amount of work done by the X component of the force exerted by the body on the water per unit length is equal to

$$-(dT_{p3}/dX)\,\delta X.$$

Equating the amount of work done by the body to the change in kinetic energy per unit length leads to

$$\rho w^* \frac{d}{dX}(Aw^*)\,\delta X - \frac{dT_{p3}}{dX}\delta X = \tfrac{1}{2}\rho \frac{d}{dX}(Aw^{*2})\,\delta X \tag{A 4}$$

and

$$\frac{dT_{\rho 3}}{dX} = \tfrac{1}{2}\rho \frac{dA}{dX} w^{*2}. \tag{A 5}$$

For the total thrust we thus obtain

$$\begin{aligned}\bar{T} = \rho \int_0^l \overline{\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(w^*A)}\,dx + \tfrac{1}{2}\rho\int_0^l \overline{w^{*2}}\frac{dA}{dx}\,dx \\ + \rho\int_0^l S\overline{\frac{\partial h}{\partial x}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi^*_{wz}}\,dx - \rho\int_0^l \overline{w\phi^*_{wz}}\frac{dS}{dx}\,dx.\end{aligned} \tag{A 6}$$

The flexible recoil mode

Upon substitution of (3.8) equations (3.9) can be written in such a way that the terms due to the presence of the waves appear on the right-hand sides only:

$$\begin{aligned}&\rho\int_0^l S\frac{\partial^2 h}{\partial t^2}dx + \rho\int_0^l \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(wA)\,dx \\ &\quad = \rho\int_0^l \left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\phi^*_{wz}A)\,dx + \rho\int_0^l S\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi^*_{wz}\,dx,\end{aligned} \tag{A 7a}$$

$$\begin{aligned}&\rho\int_0^l xS\frac{\partial^2 h}{\partial t^2}dx + \rho\int_0^l x\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(wA)\,dx \\ &\quad = \rho\int_0^l x\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\phi^*_{wz}A)\,dx + \rho\int_0^l xS\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi^*_{wz}\,dx.\end{aligned} \tag{A 7b}$$

The flexible recoil mode is obtained as a particular solution of (A 7) by equating the local rate of change of the lateral momentum of the body and the local lateral force exerted on the body by the time-varying part of the pressure of the water:

$$\rho S\frac{\partial^2 \tilde{h}}{\partial t^2} = \tilde{L}(x,t) = -\rho\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)(\tilde{w}^*A) + \rho S\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)\phi^*_{wz}, \tag{A 8}$$

where

$$\tilde{w}^* = \tilde{w} - \phi^*_{wz} = \frac{\partial \tilde{h}}{\partial t} + U\frac{\partial \tilde{h}}{\partial x} - \phi^*_{wz}.$$

Substitution of

$$\phi^*_{wz} = \alpha^* \cos\frac{2\pi}{\lambda}\{x - (U+c)\,t\}$$

and

$$\tilde{h}(x,t) = \tilde{h}_1(x)\cos\left(\frac{2\pi x}{\lambda} - \omega t\right) + \tilde{h}_2(x)\sin\left(\frac{2\pi x}{\lambda} - \omega t\right),$$

where $\omega/2\pi = |U+c|/\lambda$ is the frequency of encounter, into (A 8) and subsequently into (A 7) yields

$$\tilde{h}_1(x) = \alpha^* \frac{-S\dfrac{dA}{dx}U^2(U+2c)}{\left(\dfrac{2\pi}{\lambda}\right)^2\{S(U+c)^2+Ac^2\}^2+\left(\dfrac{dA}{dx}Uc\right)^2}, \tag{A 9a}$$

$$\tilde{h}_2(x) = \alpha^* \frac{\left(\dfrac{2\pi c}{\lambda}\right)\left[-\left(\dfrac{dA}{dx}U\right)^2-(A+S)\left(\dfrac{2\pi}{\lambda}\right)^2\{S(U+c)^2+Ac^2\}\right]}{\left(\dfrac{2\pi}{\lambda}\right)^4\{S(U+c)^2+Ac^2\}^2+\left(\dfrac{2\pi}{\lambda}\right)^2\left(\dfrac{dA}{dx}Uc\right)^2}. \tag{A 9b}$$

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